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Green's functions for gravitational waves in FRW spacetimes

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Abstract

A method for calculating the retarded Green's function for the gravitational wave equation in Friedmann-Roberson-Walker spacetimes, within the formalism of linearized Einstein gravity is developed. Hadamard's general solution to Cauchy's problem for second-order, linear partial differential equations is applied to the FRW gravitational wave equation. The retarded Green's function may be calculated for any FRW spacetime, with curved or flat spatial sections, for which the functional form of the Ricci scalar curvature R is known. The retarded Green's function for gravitational waves propagating through a cosmological fluid composed of both radiation and dust is calculated analytically for the first time. It is also shown that for all FRW spacetimes in which the Ricci scalar curvatures does not vanish, $R \neq 0$, the Green's function violates Huygens' principle; the Green's function has support inside the light-cone due to the scatter of gravitational waves off the background curvature.

I. Introduction

The Green's function is an important tool for evaluating the solution to a partial differential equation for a given set of boundary conditions. The Green's function represents the solution to Cauchy's problem: given a set of properly formulated initial data, the Green's function may be used to uniquely determine the final, evolved data. In physics, one may determine the evolution of a field according to its equations of motion, given some initial data. The techniques used in this paper have been widely applied to the calculation of Green's functions, or propagators, for quantum fields in curved spacetime. Here, those techniques will be applied to classical fields in cosmology. In this paper I will determine the evolution of a tensor field, representing gravitational waves, given a source stress-energy tensor. The goal of this paper, then, is to calculate the Green's function for gravitational waves evolving in an arbitrary FRW spacetime.

The Green's functions may be useful for the study of gravitational radiation in FRW spacetimes, especially as it relates to observational cosmology (for an example, see [1]). For situations in which the stress-energy tensor of a source of gravitational waves is known, the Green's function may be used to calculate the properties of the tensor metric perturbations, and the spectrum of gravitational radiation. Examples of such sources are cosmic strings or scalar fields present in the early universe. The tensor perturbations generated by such sources may be manifest as anisotropies in the cosmic microwave background, or as a spectrum of stochastic gravitational radiation. Then, the Green's function may be used for the calculation of physically observable quantities.

The organization of the paper is as follows. In section II, the formalism for calculating the Green's function, following Hadamard's general solution to Cauchy's problem for the wave equation, will be presented. A similar technique was developed by Waylen in [2,3]. In section III, this technique will be applied to find the Green's functions for arbitrary FRW spacetimes. The main result of this paper, a general expression for the Green's function for any FRW spacetime in which R is known, will be presented. Several specific cases will be evaluated explicitly, including the case in which the cosmological fluid undergoes a smooth transition from radiation- to dust-dominated expansion. In section IV the properties and applications of these Green's functions will be discussed. The focus will be on the cosmological significance of those terms which violate Huygens' principle. A final summary of this work will be presented in section V.

II. Formalism for calculating Green's functions

The technique for calculating Green's functions follows from Hadamard's general solution to Cauchy's problem for a second-order, linear partial differential equation [4]. The most general form for the solution to the tensor wave equation

$$h_{ab;c}{}^{c} + \Lambda h_{ab} = 0 \qquad (II.1)$$

may be written as an expansion in powers of the non-local, biscalar of geodetic interval $\sigma(x, x_i) = \frac{1}{2}s(x, x_i)^2$, where $s(x, x_i)$ is the geodetic interval between a fixed 4-vector x_i and a free 4-vector x in the spacetime. (Note that Λ is some arbitrary function, *not* the

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cosmological constant.) The Green's function solution is a bitensor, an object with tensor indices on the spatial hypersurfaces x_i and x:

$$G(x, x_i)_{ab}{}^{c_i d_i} = \frac{1}{4\pi^2}\left[u_{ab}{}^{c_i d_i}\frac{1}{\sigma} + v_{ab}{}^{c_i d_i}\log|\sigma| + w_{ab}{}^{c_i d_i}\right]. \qquad (II.2)$$

Here, u, v, and w are bitensors which are free of singularities. Extending $II.2$ into the complex-σ plane, and retaining only the imaginary part, we may obtain the retarded solution [5,6]

$$G^{ret}(x, x_i)_{ab}{}^{c_i d_i} = \frac{1}{4\pi}\left[u_{ab}{}^{c_i d_i}\delta(\sigma) - v_{ab}{}^{c_i d_i}\theta(-\sigma)\right]\theta(\lambda). \qquad (II.3)$$

The parameter λ runs between the spacetime points x_i and x, such as the time, and is positive when the spacelike hypersurface containing x lies to the future of that containing x_i. In that this equation represents the solution to the wave equation which is homogeneous everywhere except at a single point, we may obtain the solution to the inhomogeneous wave equation

$$h_{ab;c}{}^c + \Lambda h_{ab} = f_{ab} \qquad (II.4)$$

by summing the contribution of many points:

$$h(x)_{ab} = \int_{x_i}^{x}\sqrt{-g(x')}d^4x' G^{ret}(x, x')_{ab}{}^{c_i d_i} f(x')_{c_i d_i}. \qquad (II.5)$$

Thus, the Green's function may be used to construct solutions to the wave equation. Equation $II.3$, then, gives the retarded Green's function which we seek.

A prescription for calculating the non-singular, bitensor quantities u and v may be obtained by generalizing the work of DeWitt and Brehme [5,6] and McLenaghan [7] on the scalar and vector wave equations. Such a calculation has been carried out by Waylen [2,3]. One may apply Hadamard's general solution $II.2$ to the homogenous wave equation, and equate terms with matching powers of σ. One obtains

$$
\begin{aligned}
(\sigma_{;c}{}^c - 4)u_{ab}{}^{c_i d_i} + 2u_{ab}{}^{c_i d_i}{}_{;c}\sigma^{;c} &= 0\\
2\sigma_{;c}v_{ab}{}^{c_i d_i;c} + (\sigma_{;c}{}^c - 2)v_{ab}{}^{c_i d_i} + u_{ab}{}^{c_i d_i}{}_{;c}{}^c + \Lambda u_{ab}{}^{c_i d_i} &= 0.
\end{aligned} \qquad (II.6)
$$

Manipulating these equations, we find that [7]

$$
\begin{aligned}
\ln u &= -\frac{1}{2}\int\frac{d\lambda}{\lambda}(\sigma_{;c}{}^c - 4)\\
v_{ab}{}^{c_i d_i} &= -\frac{1}{2\lambda}u_{ab}{}^{c_i d_i}\int d\lambda(u_{ef}{}^{g_i h_i}{}_{;c}{}^c + \Lambda u_{ef}{}^{g_i h_i})(u^{-1})^{ef}{}_{g_i h_i}.
\end{aligned} \qquad (II.7)
$$

In general, the bitensors u and v will be proportional to the bitensor of parallel geodetic transport (see the discussion of such bitensors in [5]). The functions u and v may be

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evaluated once the biscalar of geodetic interval σ is known. Therefore, for a given metric, σ, u, and v may be determined, and the Green's function for the tensor wave equation may be evaluated. This general technique may be applied to the wave equation in any spacetime.

III. Green's functions in FRW spacetimes

The gravitational wave equation for which the Green's function will be determined, describing the propagation of tensor metric perturbations on a background spacetime, may be derived in the context of linearized, Einstein gravity. (For some details on linearized, Einstein gravity, see [8,9]). I will consider tensor metric perturbations, $g_{ab} \rightarrow g_{ab} + h_{ab}$, where the line element for the background, FRW metric is

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + \frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right). \qquad (III.1)$$

Due to the symmetries of the FRW spacetime, I may apply the transverse, traceless, synchronous (TTS) gauge conditions [10,11]

$$h_{ab}{}^{;b} = h_{ab}g^{ab} = h_{ab}t^b = 0 \qquad (III.2)$$

to isolate the physical degrees of freedom in the gravitational radiation. Here, t_a is a unit 4-vector pointing in the time direction η, orthogonal to spacelike hypersurfaces. Ultimately, the gravitational wave equation is found to be

$$16\pi G T_{ab}^{source} = h_{ab;c}{}^c + 2R^c{}_a{}^d{}_b h_{cd}. \qquad (III.3)$$

Here, $R^c{}_a{}^d{}_b$ is the Riemann curvature tensor in the background FRW spacetime, and G is the gravitational contant. (We use units such that the speed of light is unity: $c = 1$.) The stress-energy tensor T_{ab}^{source} represents the external source of the gravitational waves.

We may now apply the techniques developed in the previous section in order to evaluate the retarded Green's function. The bitensor functions u and v are proportional to the transverse, traceless, synchronous bitensor $\Sigma(x, x_i)_{ab}{}^{c_i d_i}$, which projects only the TTS portion of the source stress-energy tensor from the spacelike hypersurface at x_i to x. The TTS bitensor $\Sigma_{ab}{}^{c_i d_i}$ may be constructed through the use of the spatial biscalar of geodetic interval, $\mu(\vec{x}, \vec{x_i}) = |\vec{x} - \vec{x_i}|$, following the work of Allen and Jacobson [12,13]. Defining

$$\begin{aligned} V_a &= \nabla_a \mu(\vec{x}, \vec{x_i}) & P_{ab} &= g_{ab} + t_a t_b \\ V_{a_i} &= \nabla_{a_i} \mu(\vec{x}, \vec{x_i}) & P_{ab_i} &= g^b{}_{b_i} P_{ab} \end{aligned} \qquad (III.4)$$

the unique TTS bitensor with indices a, b on the tangent space at x, and c_i, d_i at x_i is

$$\begin{aligned} \Sigma_{ab}{}^{c_i d_i} =\; &P_a{}^{c_i} P_b{}^{d_i} + P_a{}^{d_i} P_b{}^{c_i} - P_{ab} P^{c_i d_i} + V_a V_b P^{c_i d_i} + P_{ab} V^{c_i} V^{d_i} \\ &+ P_a{}^{c_i} V_b V^{d_i} + P_a{}^{d_i} V_b V^{c_i} + P_b{}^{c_i} V_a V^{d_i} + P_b{}^{d_i} V_a V^{c_i} + V_a V_b V^{c_i} V^{d_i}. \end{aligned} \qquad (III.5)$$

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Here, $V^a \Sigma_{ab}{}^{c_i d_i} = g^{ab} \Sigma_{ab}{}^{c_i d_i} = t^a \Sigma_{ab}{}^{c_i d_i} = 0$, such that it is transverse, traceless, and synchronous. Thus, combining equations $II.3, 7$, and $III.3$ the entire prescription for calculating the Green's function for tensor metric perturbations is specified.

The calculation of the Green's function for FRW spacetimes may now be carried out. Having specified the background metric, the biscalar of geodetic interval is $\sigma = \frac{1}{2}(|\vec{x} - \vec{x}_i|^2 - (\eta - \eta_i)^2)$. Evaluating equations $II.7$,

$$u_{ab}{}^{c_i d_i} = \frac{C(|\vec{x} - \vec{x}'|/\mathcal{R})}{a(\eta_i)a(\eta)} \Sigma_{ab}{}^{c_i d_i}$$

$$v_{ab}{}^{c_i d_i} = -\frac{u_{ab}{}^{c_i d_i}}{12(\eta - \eta_i)} \int_{\eta_i}^{\eta} a^2(\eta) R d\eta. \qquad (III.6)$$

In this equation $C(x) = x/\sin(x)$ and $\mathcal{R}$ is the curvature scale of the spatial sections of the spacetime such that ${}^3 R = 6\mathcal{R}^{-2}$. In the case $K = 0$, $\mathcal{R} \to \infty$ so that $C = 1$. Applying the above results to equations $II.3$ and $III.3$, one may write the Green's function as

$$G^{ret}(x, x_i)_{ab}{}^{c_i d_i} = \frac{C(|\vec{x} - \vec{x}'|/\mathcal{R})}{4\pi a(\eta_i)a(\eta)} \left[\frac{\delta(\eta - \eta_i - |\vec{x} - \vec{x}_i|)}{|\vec{x} - \vec{x}_i|} \right.$$
$$\left. + \frac{1}{12} \Big(\int_{\eta_i}^{\eta} a^2(\eta) R d\eta \Big) \frac{\theta(\eta - \eta_i - |\vec{x} - \vec{x}_i|)}{\eta - \eta_i} \right] \theta(\eta - \eta_i) \Sigma(x, x_i)_{ab}{}^{c_i d_i}. \qquad (III.7)$$

Note that the Green's function is symmetric under interchange $x \leftrightarrow x_i$, as would be expected. Also, the Green's function possesses the correct dimensionality, (length)$^{-2}$, where the coordinates $(\eta, \vec{x})$ are dimensionless and the expansion scale factor a carries units of length. This equation composes the main result of this paper.

The Green's function may be evaluated for a number of specific cases. The results are displayed in the accompanying table. (*See table.*) In this table, $\alpha = -1, 1, 2$ for deSitter spacetime, radiation- and dust-dominated expansion, respectively. Here, the equation of state of the cosmological fluid is $p = \frac{2-\alpha}{3\alpha}\rho$. Minkowski spacetime is given by $K = 0$ and $a(\eta) = 1$ (for which case the preceding equation of state is invalid). The first three entries in the table give the expansion scale factor and the Green's function for an ideal cosmological fluid, in a spacetime with flat and curved spatial sections. The results from the first entry, for flat spatial sections, have been previously calculated for the specific cases $\alpha = -1, 1, 2$ [1,14]. The Green's function for arbitrary α and for any K is given here for the first time. The last three entries in the table give the expansion scale factor and Green's function for a mixed cosmological fluid, in a spacetime with flat and curved spatial sections. In the case of the mixed ideal fluid, the energy density is $\rho = \rho_{rad} + \rho_{mat}$ and the pressure is $p = \frac{1}{3}\rho_{rad}$, where τ is a function of $\rho_{crit}(\eta_0)$ and $\rho_{rad}(\eta_0)$, and η_0 is the present-day conformal time. Then, for $\eta \ll \tau$, the fluid is radiation-dominated. For $\eta \gg \tau$, the fluid is dust-dominated. The Green's function for this mixed case is presented here for the first time. This case represents the most realistic cosmological scenario.

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IV. Properties and applications of Green's functions

The Green's functions describing the evolution of gravitational radiation in FRW spacetimes, derived in the preceding section, display interesting properties and have useful applications, which may now be discussed. Specifically, one may examine the physical consequences of the terms in the Green's functions which contribute to the violation of Huygens' principle. Applications of the analytic Green's functions for open and closed spacetimes, for a combined radiation and matter fluid will be discussed.

Huygens' principle states that the effect of a luminous disturbance about some point at time t will be localized at a later time t' in a very thin spherical shell at a radius $c(t'-t)$ [4]. So, a wave packet travels spherically outward on the light cone from some source. From examining equation $II.3$, one finds that for $v \neq 0$, the Green's function has support inside the light cone; the "wave packet" disperses as it travels and is not localized on a thin spherical shell. In an FRW spacetime, equation $III.7$ indicates that when $R \neq 0$ Huygens' principle is violated.

The violation of Huygens' principle is due to the coupling of the curvature of spacetime to the tensor metric perturbations. The energy in the background curvature feeds into the gravitational radiation. This coupling of spacetime curvature to fields has been well studied on the microscopic level in the context of particle creation [15,16]. (See references [17,18] for a discussion of quantum mechanical propagators with respect to Huygens' principle.) A classical analogue of particle creation is observed here on the macroscopic level [19,20]. Examining the table, the function V increases with the expansion $a(\eta)$; V represents the amplification of the classical gravitational field due to the expansion. One may say that energy is transferred from the expansion to the gravitational field. Ultimately, it may be simply stated that Huygens' principle for gravitational waves propagating in an FRW spacetime is satisfied only when $R = 0$, or the spacetime is filled with a conformally-invariant radiation fluid.

The Green's function for the case of the mixed radiation-plus-dust fluid, for $K = \pm 1, 0$, as presented in the table, is new. These analytic expressions should be useful for examining the evolution of gravitational waves through the transition from radiation- to dust-dominated expansion.

The primary application of the Green's functions derived in this paper is for the situation in which the source stress-energy tensor is known. The source ought to be "stiff", in that the source evolves freely of the background spacetime and the perturbations it produces. Some examples are cosmic strings, global topological defects, and scalar fields [1,14,21,22]. Of interest are the anisotropies produced in the microwave background, and the spectrum of gravitational radiation emitted.

The anisotropies in the microwave background may be calculated by summing the contributions to the temperature fluctuations caused by the perturbations in the gravitational field along the path length of a photon traveling from the surface of last scattering.

$$
\begin{aligned}
\frac{\delta T}{T} &= -\frac{1}{2} \int d\lambda\, e^a(\lambda) e^b(\lambda) \frac{\partial}{\partial \eta} \Big(\frac{1}{a^2(\eta)} h_{ab} \Big) \\
&= -\frac{1}{2} \int d\lambda\, e^a(\lambda) e^b(\lambda) \frac{\partial}{\partial \eta} \Big(\frac{1}{a^2(\eta)} \int \sqrt{-g_i}\, d^4 x_i\, G^{ret}(x, x_i)_{ab}{}^{c_i d_i} 16\pi G T^{source}(x_i)_{c_i d_i} \Big)
\end{aligned}
$$

$$(IV.1)$$

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For a given source stress-energy tensor, the temperature fluctuations $\delta T/T$ causedd by cosmic strings, global topological defects, or scalar fields [1,21,14,22] may be calculated.

The energy density and power in the gravitational radiation emitted by a given source may be examined by calculating the stress-energy tensor

$$8\pi G T_{ab}^{grav} = \frac{1}{2} h^{cd} \left[h_{da;bc} + h_{db;ac} - h_{ab;cd} - h_{cd;ab} \right] - \frac{1}{4} h^c{}_{d;a} h^d{}_{c;b}$$
$$+ \frac{1}{2} g_{ab} \left[h^{cd} h_{cd;n}{}^n + \frac{1}{4} h_{cd;n} h^{cd;n} - h^{cd} h_{cd} \left(\frac{K}{a^2} + \frac{a'^2}{a^4} \right) \right]. \qquad (IV.2)$$

This stress-energy tensor, the terms occuring in the perturbed Einstein's equations which are second-order in h, may be evaluated for a given source through the use of the Green's function. This expression may have use in studying the power in tensor metric perturbations produced during inflation (for a recent review, see [23]), or through the collision of bubbles formed in a first-order phase transition [24]. In this case, the Green's functions derived in the preceding section for the mixed ideal fluid may be especially useful for calculating the evolution of the gravitational waves through the transition from radiation- to dust-dominated expansion.

Finally, it is interesting to note that while the Green's function for the mixed ideal fluid has been obtained, the mode functions for the gravitational waves in such a case apparently cannot be written in terms of known functions. In the case of a single, ideal fluid background, the mode functions, solutions to equation $II.1$, take the form of Bessel functions. The case of the mixed fluid, however, appears much more complicated.

V. Conclusion

In this paper the retarded Green's functions for the gravitational wave equation in Friedmann-Robertson-Walker spacetimes, within the formalism of linearized Einstein gravity, were calculated. While the form for the Green's function for a generic FRW spacetime was presented, several specific cases were considered. These cases, presented for the first time, include an ideal cosmological fluid with an equation of state $p = \frac{2-\alpha}{3\alpha}\rho$, and the case of a mixed ideal fluid of radiation and collisionless dust, where $\rho = \rho_{rad} + \rho_{mat}$ and $p = \frac{1}{3}\rho_{rad}$. The Green's functions for varying spatial curvature, $K = 0, \pm 1$ were also considered. It was also shown that for all non-conformally invariant FRW spacetimes, in which $R \neq 0$, the Green's function violates Huygens' principle. This is the classical analogue of particle creation in a varying gravitational field, as the gravitational waves scatter off the background curvature and gain energy from the cosmological expansion. Finally, it was indicated how these Green's functions may be applied to the calculation of the microwave anisotropies and spectrum of gravitational radiation produced by "stiff" sources. These applications of the Green's functions will be carried out in a future work.

Table

Green's functions for gravitational waves in FRW spacetimes				
$G^{ret}(x,x_i)_{ab}{}^{c_i d_i} = \frac{C(|\vec{x}-\vec{x}_i|/\mathcal{R})}{4\pi a(\eta_i)a(\eta)}\left[\frac{\delta(\eta-\eta_i-|\vec{x}-\vec{x}_i|)}{|\vec{x}-\vec{x}_i|} + V\frac{\theta(\eta-\eta_i-|\vec{x}-\vec{x}_i|)}{\eta-\eta_i}\right]\theta(\eta-\eta_i)\Sigma(x,x_i)_{ab}{}^{c_i d_i}$				
K	$C(|\vec{x}-\vec{x}_i|/\mathcal{R})$	$a(\eta)$	$V(\eta)$	**cosmology**
0	1	$a(\eta_i)[\frac{\eta}{\eta_i}]^\alpha$	$\frac{\alpha(\alpha-1)(\eta-\eta_i)}{2\eta_i\eta}$	ideal fluid
-1	$\frac{|\vec{x}-\vec{x}_i|}{\mathcal{R}}/\sinh(\frac{|\vec{x}-\vec{x}_i|}{\mathcal{R}})$	$a(\eta_i)[\frac{\sinh(\eta/\alpha)}{\sinh(\eta_i/\alpha)}]^\alpha$	$\frac{1}{2}(\alpha-1)\left(\frac{\sinh[(\eta-\eta_i)/\alpha]}{\sinh(\eta/\alpha)\sinh(\eta_i/\alpha)}\right)$	ideal fluid
1	$\frac{|\vec{x}-\vec{x}_i|}{\mathcal{R}}/\sin(\frac{|\vec{x}-\vec{x}_i|}{\mathcal{R}})$	$a(\eta_i)[\frac{\sin(\eta/\alpha)}{\sin(\eta_i/\alpha)}]^\alpha$	$\frac{1}{2}(\alpha-1)\left(\frac{\sin[(\eta-\eta_i)/\alpha]}{\sin(\eta/\alpha)\sin(\eta_i/\alpha)}\right)$	ideal fluid
0	1	$a(\eta_i)[\frac{\eta(\eta+\tau)}{\eta_i(\eta_i+\tau)}]$	$\frac{1}{\tau}\log\frac{\eta(\eta_i+\tau)}{\eta_i(\eta+\tau)}$	mixed fluid
-1	$\frac{|\vec{x}-\vec{x}_i|}{\mathcal{R}}/\sinh(\frac{|\vec{x}-\vec{x}_i|}{\mathcal{R}})$	$a(\eta_i)[\frac{\tau\sinh\eta+\sinh^2(\eta/2)}{\tau\sinh\eta_i+\sinh^2(\eta_i/2)}]$	$\frac{1}{4\tau}\log\left[\frac{2\tau\coth(\eta_i/2)+1}{2\tau\coth(\eta/2)+1}\right]$	mixed fluid
1	$\frac{|\vec{x}-\vec{x}_i|}{\mathcal{R}}/\sin(\frac{|\vec{x}-\vec{x}_i|}{\mathcal{R}})$	$a(\eta_i)[\frac{\tau\sin\eta+\sin^2(\eta/2)}{\tau\sin\eta_i+\sin^2(\eta_i/2)}]$	$\frac{1}{4\tau}\log\left[\frac{2\tau\cot(\eta_i/2)+1}{2\tau\cot(\eta/2)+1}\right]$	mixed fluid

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